

Normality and quotient in crossed modules, cat^1 -groups and internal groupoids within groups with operations

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Abstract

In this paper we define the notions of normal subcrossed module and quotient crossed module within groups with operations; and using the equivalence of crossed modules over groups with operations and internal groupoids we prove how normality and quotient concepts are related in these two categories. Further we prove an equivalence of crossed modules over groups with operations and cat^1 -groups with operations for a certain algebraic category; and then by this equivalence we determine normal and quotient objects in the category of cat^1 -groups with operations. Finally we characterize the coverings of cat^1 -groups with operations.

Key Words: Group with operations, quotient crossed module, internal groupoid.

Classification: 20L05, 57M10, 22AXX, 22A22

1 Introduction

Crossed modules as defined by Whitehead [30, 31] have been widely used in homotopy theory [7], the theory of group representation (see [9] for a survey), in algebraic K-theory [20], and homological algebra [19, 22]. Crossed modules can be viewed as 2-dimensional groups [5].

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The notions of subcrossed module and normal subcrossed module were defined in [28]. In [11] Brown and Spencer proved that the category of internal groupoids within the groups, which are also called in [11] under the name of \mathcal{G} -groupoids and alternative names, quite generally used are *group-groupoid* [10] or *2-group* (see for example [3]) is equivalent to the category of crossed modules of groups. Using the equivalence in [11], recently in [24] normal and quotient objects in the category of group-groupoids have been obtained.

In [21] Loday defined an algebraic object called *cat¹-group* as a group G with two endomorphisms s, t of G such that $st = t, ts = s$ and $[Kers, Kert] = 0$, where $[Kers, Kert]$ represents the commutator subgroup of G ; and proved that the categories of cat¹-groups and crossed modules are equivalent.

In [29] Porter proved a similar result to one in [11] holds for certain algebraic categories, introduced by Orzech [27], which definition was adapted by him and called category of groups with operations. Applying Porter's result, the study of internal category theory was continued in the works of Datuashvili [15] and [16]. Moreover, she developed cohomology theory of internal categories, equivalently, crossed modules, in categories of groups with operations [13] and [14]. The equivalences of the categories in [11] and [29] enable us to generalize some results on group-groupoids which are internal categories within groups to the more general internal groupoids for a certain algebraic category \mathcal{C} (see for example [2], [23], [25] and [26]).

In this paper for an algebraic category \mathcal{C} we define normal subcrossed module and quotient crossed module for groups with operations; and then obtain normal subgroupoid and quotient groupoid in internal groupoids corresponding respectively to a normal subcrossed module and a quotient crossed module of groups with operations. Further we prove an equivalence of crossed modules over groups with operations and cat¹-groups with operations for a certain algebraic category. This equivalence enables us to determine normal and quotient objects; and coverings in the category of cat¹-groups with operations.

Main results of this paper constitute some parts of the PhD thesis of first author at Erciyes University in 2014.

2 Preliminaries

Let G be a groupoid. We write G_0 for the set of objects of G and write G_1 for the set of morphisms. We also identify G_0 with the set of identities of G and so an element of G_0 may be written as x or 1_x as convenient. We write $d_0, d_1: G_1 \rightarrow G_0$ for the source and target maps, and, as usual, write $G(x, y)$ for $d_0^{-1}(x) \cap d_1^{-1}(y)$, for $x, y \in G_0$. The composition $h \circ g$ of two elements of G is defined if and only if $d_0(h) = d_1(g)$, and so the map $(h, g) \mapsto h \circ g$

is defined on the pullback $G_{1d_0} \times_{d_1} G_1$ of d_0 and d_1 . The *inverse* of $g \in G(x, y)$ is denoted by $g^{-1} \in G(y, x)$.

If $x \in G_0$, we write $\text{St}_G x$ for $d_0^{-1}(x)$ and call the *star* of G at x . Similarly we write $\text{Cost}_G x$ for $d_1^{-1}(x)$ and call *costar* of G at x . The set of all morphisms from x to x is a group, called *object group* at x , and denoted by $G(x)$.

A groupoid G is *totally intransitive* if $G(x, y) = \emptyset$ for all $x, y \in G_0$ such that $x \neq y$. Such a groupoid is determined entirely by the family $\{G(x) \mid x \in G_0\}$ of groups. This totally intransitive groupoid is sometimes called *totally disconnected* or *bundle of groups* (Brown [4, pp.218]).

Let G be a groupoid. A *subgroupoid* H of G is a pair of subsets $H_1 \subseteq G_1$ and $H_0 \subseteq G_0$ such that $d_0(H_1) \subseteq H_0$, $d_1(H_1) \subseteq H_0$, $1_x \in H_1$ for each $x \in H_0$ and H_1 is closed under the partial multiplication and the inversion in G . A subgroupoid H of G is called *wide* if $H_0 = G_0$.

Definition 2.1 Let G be a groupoid. A subgroupoid N of G is called *normal* if it is wide in G and $g \circ N(x) = N(y) \circ g$ for objects $x, y \in G_0$ and $g \in G(x, y)$.

Quotient groupoid is formed as follows (Higgins [17, pp.86] and Brown [4, pp.420]). Let N be a normal subgroupoid of the groupoid G . The components of N define a partition on G_0 and we write $[x]$ for the class containing x . Then N also defines an equivalence relation on G_1 by $g \sim h$ for $a, b \in G_1$ if and only if $g = n \circ h \circ m$ for some $m, n \in N_1$. A partial composition $[h] \circ [g]$ on the morphisms is defined if and only if there exist $g_1 \in [g]$, $h_1 \in [h]$ such that $h_1 \circ g_1$ is defined in G_1 and then $[h] \circ [g] = [h_1 \circ g_1]$. This partial composition defines a groupoid on classes $[x]$'s as objects. The groupoid defined in this manner is called *quotient groupoid* and denoted by G/N .

As it is stated in Brown [4, pp.218], in the case where the normal subgroupoid N is totally intransitive, we have that $(G/N)_0 = G_0$ and $G/N(x, y)$ consists of all cosets $g \circ N(x)$ for $x, y \in G_0$ and $g \in G(x, y)$. The groupoid composition becomes

$$(h \circ N(y)) \circ (g \circ N(x)) = (h \circ g) \circ N(x).$$

for $g \in G(x, y)$ and $h \in G(y, z)$.

We recall that a crossed module of groups originally defined by Whitehead [30, 31], consists of two groups A and B , an action of B on A denoted by $b \cdot a$ for $a \in A$ and $b \in B$; and a morphism $\alpha: A \rightarrow B$ of groups satisfying the following conditions for all $a, a_1 \in A$ and $b \in B$

$$(i) \quad \alpha(b \cdot a) = b + \alpha(a) - b,$$

$$(ii) \quad \alpha(a) \cdot a_1 = a + a_1 - a.$$

We will denote such a crossed module by (A, B, α) . Let (A, B, α) and (A', B', α') be two crossed modules. A morphism (f_1, f_2) from (A, B, α) to (A', B', α') is a pair of morphisms of groups $f_1: A \rightarrow A'$ and $f_2: B \rightarrow B'$ such that $f_2\alpha = \alpha'f_1$ and $f_1(b \cdot a) = f_2(b) \cdot f_1(a)$ for $a \in A$ and $b \in B$.

It was proved by Brown and Spencer in [11, Theorem 1] that the category $X\text{Mod}(\text{Grp})$ of crossed modules over groups is equivalent to the category GrpGpd of group-groupoids.

3 Normal and quotient crossed modules in groups with operations

The idea of the definition of categories of groups with operations comes from Higgins [18] and Orzech [27]; and the definition below is from Porter [29] and Datuashvili [12, pp. 21], which is adapted from Orzech [27].

Definition 3.1 From now on \mathcal{C} will be a category of groups with a set of operations Ω and with a set E of identities such that E includes the group laws, and the following conditions hold: If Ω_i is the set of i -ary operations in Ω , then

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(b) The group operations written additively 0 , $-$ and $+$ are respectively the elements of Ω_0 , Ω_1 and Ω_2 . Let $\Omega'_2 = \Omega_2 \setminus \{+\}$, $\Omega'_1 = \Omega_1 \setminus \{-\}$ and assume that if $\star \in \Omega'_2$, then \star° defined by $a \star^\circ b = b \star a$ is also in Ω'_2 . Also assume that $\Omega_0 = \{0\}$;

(c) For each $\star \in \Omega'_2$, E includes the identity $a \star (b + c) = a \star b + a \star c$;

(d) For each $\omega \in \Omega'_1$ and $\star \in \Omega'_2$, E includes the identities $\omega(a + b) = \omega(a) + \omega(b)$ and $\omega(a) \star b = \omega(a \star b)$.

A category satisfying the conditions (a)-(d) is called a *category of groups with operations*.

Remark 3.2 The set Ω_0 contains exactly one element, the group identity; hence for instance the category of associative rings with unit is not a category of groups with operations.

Example 3.3 The categories of groups, rings generally without identity, R -modules, associative, associative commutative, Lie, Leibniz, alternative algebras are examples of categories of groups with operations.

A *morphism* between any two objects of \mathcal{C} is a group homomorphism, which preserves the operations in Ω'_1 and Ω'_2 .

The topological version of this definition can be stated as follows:

Definition 3.4 Let X be an object in C . If X has a topology such that all operations in Ω are continuous, then X is called a *topological group with operations* in C .

In particular if C is the category of groups, then a topological group with operations just becomes a topological group and if C is the category of R -modules, then it becomes a topological R -module.

We will denote the category of topological groups with operations by Top^C .

For the objects A and B of C , the direct product $A \times B$ with the usual operations becomes a group with operations. Hence the category C has finite products.

The subobject in the category C can be defined as follows.

Definition 3.5 Let A be an object in C . A subset $B \subseteq A$ is called a *subgroup with operations* of A if the following conditions are satisfied:

- (i) $b \star b_1 \in B$ for $b, b_1 \in B$ and $\star \in \Omega_2$;
- (ii) $\omega(b) \in B$ for $b \in B$ and $\omega \in \Omega_1$.

The normal subobject in the category C of groups with operations is defined as follows.

Definition 3.6 [27, Definition 1.7] Let A be an object in C and N a subgroup with operations of A . N is called a *normal subgroup with operations* or an *ideal* of A and written $N \triangleleft A$ if the following conditions are satisfied:

- (i) $(N, +)$ is a normal subgroup of $(A, +)$;
- (ii) $a \star n \in N$ for $a \in A, n \in N$ and $\star \in \Omega'_2$.

For a morphism $f: A \rightarrow B$ in C , $\text{Ker } f = \{a \in A \mid f(a) = 0\}$ is an ideal of A .

A quotient object in C is constructed as follows: Let A be an object in C and N an ideal of A . Then the relation on A defined by

$$a \sim a_1 \quad \text{iff} \quad a - a_1 \in N$$

is an equivalence relation. Then the quotient set A/N along with the operations defined by

$$\begin{aligned} [a] \star [a_1] &= [a \star a_1] \\ \omega([a]) &= [\omega(a)] \end{aligned}$$

for $\star \in \Omega_2, \omega \in \Omega_1$ becomes an object in C and called *quotient group with operations of A by N* .

In the following proposition we prove that the category C has kernels in categorical sense.

Proposition 3.7 *Let A be an object in \mathcal{C} . Then N is an ideal of A if and only if it is a kernel of a morphism in \mathcal{C} .*

Proof: We have already seen that the kernel of a morphism in \mathcal{C} is an ideal of the domain.

Conversely if N is an ideal of A , then the quotient A/N becomes an object in \mathcal{C} and quotient morphism $p: A \rightarrow A/N$ has N as kernel. ■

Let A and B be two groups with operations in \mathcal{C} . An *extension* of A by B is an exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{p} B \longrightarrow 0 \quad (1)$$

in which p is surjective and ι is the kernel of p . It is *split* if there is a morphism $s: B \rightarrow E$ such that $ps = \text{id}_B$. A split extension of B by A is called a *B-structure* on A . Given such a *B-structure* on A we get actions of B on A corresponding to the operations in \mathcal{C} . For any $b \in B$, $a \in A$ and $\star \in \Omega'_2$ we have the actions called *derived actions* by Orzech [27, pp.293]

$$\begin{aligned} b \cdot a &= s(b) + a - s(b) \\ b \star a &= s(b) \star a. \end{aligned} \quad (2)$$

Given a set of actions of B on A (one for each operation in Ω_2), let $A \rtimes B$ be a universal algebra whose underlying set is $A \times B$ and whose operations are

$$\begin{aligned} (a', b') + (a, b) &= (a' + b' \cdot a, b' + b), \\ (a', b') \star (a, b) &= (a' \star a + a' \star b + b' \star a, b' \star b). \end{aligned}$$

Theorem 3.8 [27, Theorem 2.4] *A set of actions of B on A is a set of derived actions if and only if $A \rtimes B$ is an object of \mathcal{C} .*

We recall that for groups with operations A and B , in [13, Proposition 1.1] all necessary and sufficient conditions for the actions of B on A to be derived actions are determined.

Lemma 3.9 *Let A, B, S and T be objects in \mathcal{C} . Suppose that we have a set of derived actions of B on A and a set of derived actions of T on S . If $S \rtimes T$ is an ideal of $A \rtimes B$ then the followings are satisfied.*

- (a) *S and T are ideals of A and B , respectively,*
- (b) *$b \cdot s \in S$ for all $b \in B, s \in S$,*
- (c) *$(t \cdot a) - a \in S$ for all $t \in T, a \in A$,*
- (d) *$b \star s \in S$ for all $b \in B, s \in S$,*

(e) $t \star a \in S$ for all $t \in T, a \in A$.

Proof: In the following proofs we assume that $S \rtimes T$ is an ideal of $A \rtimes B$ and $\star \in \Omega'_2$.

(a) For $(s, 0) \in S \rtimes T$ and $(a, 0) \in A \rtimes B$, we have

$$(a, 0) + (s, 0) - (a, 0) = (a + s - a, 0) \in S \rtimes T$$

and

$$(a, 0) \star (s, 0) = (a \star s, 0) \in S \rtimes T.$$

Hence $a + s - a \in S$ and S is an ideal of A . Similarly if $(0, t) \in S \rtimes T$ and $(0, b) \in A \rtimes B$, then

$$(0, b) + (0, t) - (0, b) = (0, b + t - b) \in S \rtimes T$$

and

$$(0, b) \star (0, t) = (0, b \star t) \in S \rtimes T.$$

Hence T is an ideal of B .

(b) If $(s, 0) \in S \rtimes T$ and $(0, b) \in A \rtimes B$, then

$$(0, b) + (s, 0) - (0, b) = (b \cdot s, 0) \in S \rtimes T$$

and hence $b \cdot s \in S$.

(c) For $(0, -t) \in S \rtimes T$ and $(t \cdot a, 0) \in A \rtimes B$, it implies that

$$(t \cdot a, 0) + (0, -t) - (t \cdot a, 0) = ((t \cdot a) - a, -t) \in S \rtimes T$$

and so $(t \cdot a) - a \in S$.

(d) For $(s, 0) \in S \rtimes T$ and $(0, b) \in A \rtimes B$, we have

$$(0, b) \star (s, 0) = (b \star s, 0) \in S \rtimes T$$

and therefore $b \star s \in S$.

(e) If $(0, t) \in S \rtimes T$ and $(a, 0) \in A \rtimes B$, then

$$(0, t) \star (a, 0) = (t \star a, 0) \in S \rtimes T.$$

and $t \star a \in S$.

■

Let \mathcal{E} be a B -structure on A and let \mathcal{F} be a T -structure on S as below

$$\mathcal{E} : \quad 0 \longrightarrow A \xrightarrow{\iota} E \xrightleftharpoons[p]{s} B \longrightarrow 0$$

$$\mathcal{F} : \quad 0 \longrightarrow S \xrightarrow{\iota'} F \xrightleftharpoons[p']{s'} T \longrightarrow 0.$$

If F is an ideal of E then \mathcal{F} is a normal substructure of \mathcal{E} . Then we can construct the quotient split extension as follows.

$$\mathcal{E}/\mathcal{F} : \quad 0 \longrightarrow A/S \xrightarrow{\iota^*} E/F \xrightleftharpoons[p^*]{s^*} B/T \longrightarrow 0$$

That is, \mathcal{E}/\mathcal{F} is a B/T -structure on A/S .

Definition 3.10 [29] A crossed module in \mathbf{C} is a triple (A, B, α) , where A and B are the objects of \mathbf{C} , B acts on A , i.e., we have a derived action in \mathbf{C} , and $\alpha: A \rightarrow B$ is a morphism in \mathbf{C} with the conditions:

$$\text{CM1. } \alpha(b \cdot a) = b + \alpha(a) - b;$$

$$\text{CM2. } \alpha(a) \cdot a' = a + a' - a;$$

$$\text{CM3. } \alpha(a) \star a' = a \star a';$$

$$\text{CM4. } \alpha(b \star a) = b \star \alpha(a), \alpha(a \star b) = \alpha(a) \star b$$

for any $b \in B, a, a' \in A$, and $\star \in \Omega'_2$.

A morphism $(A, B, \alpha) \rightarrow (A', B', \alpha')$ between two crossed modules is a pair $f: A \rightarrow A'$ and $g: B \rightarrow B'$ of the morphisms in \mathbf{C} such that

$$(i) \quad g\alpha(a) = \alpha'f(a),$$

$$(ii) \quad f(b \cdot a) = g(b) \cdot f(a),$$

$$(iii) \quad f(b \star a) = g(b) \star f(a)$$

for any $b \in B, a \in A$ and $\star \in \Omega'_2$.

Definition 3.11 We call a crossed module (S, T, σ) in \mathbf{C} as a *subcrossed module* of a crossed module (A, B, α) in \mathbf{C} if

SCM1. S is a subobject of A ; and T is a subobject of B ;

SCM2. σ is the restriction of α to S ;

SCM3. the action of T on S is induced by the action of B on A .

Definition 3.12 A subcrossed module (S, T, σ) of (A, B, α) in the sense of Definition 3.11 is called *normal* if

NCM1. T is an ideal of B ,

NCM2. $b \cdot s \in S$ for all $b \in B, s \in S$,

NCM3. $(t \cdot a) - a \in S$ for all $t \in T, a \in A$,

NCM4. $b \star s \in S$ for all $b \in B, s \in S$,

NCM5. $t \star a \in S$ for all $t \in T, a \in A$.

Remark 3.13 Here we note that if (S, T, σ) is a normal subcrossed module of (A, B, α) then by the conditions [CM2] of Definition 3.10; and [NCM2] and [NCM4] of Definition 3.12, S becomes an ideal of A .

As an example if $(f, g): (A, B, \alpha) \rightarrow (A', B', \alpha')$ is a morphism of crossed modules in \mathbf{C} , then $(\text{Ker} f, \text{Ker} g, \alpha|_{\text{Ker} f})$, the kernel of (f, g) , is a normal subcrossed module of (A, B, α) .

As a corollary of Lemma 3.9 a normal crossed module in \mathbf{C} can be characterized as follow.

Corollary 3.14 A subcrossed module (S, T, σ) of (A, B, α) is normal if and only if $S \rtimes T$ is an ideal of $A \rtimes B$.

We now obtain quotient crossed module in \mathbf{C} as follows:

Theorem 3.15 Let (S, T, σ) be a normal subcrossed module of (A, B, α) in \mathbf{C} . Then we have a crossed module $(A/S, B/T, \alpha^*)$ called quotient crossed module where A/S and B/T are quotient groups with operations.

Proof: The actions of B/T on A/S are defined by

$$[b] \cdot [a] = [b \cdot a]$$

$$[b] \star [a] = [b \star a].$$

These actions are well defined. If $[b] = [b_1] \in B/T$ and $[a] = [a_1] \in A/S$, then $b - b_1 \in T$ and $a - a_1 \in S$. Hence

$$\begin{aligned} b \cdot a - b_1 \cdot a_1 &= b \cdot (a - a_1 + a_1) - b_1 \cdot a_1 \\ &= b \cdot (a - a_1) + b \cdot a_1 - b_1 \cdot a_1 \\ &= b \cdot (a - a_1) + (b - b_1 + b_1) \cdot a_1 - b_1 \cdot a_1 \\ &= b \cdot (a - a_1) + (b - b_1) \cdot (b_1 \cdot a_1) - b_1 \cdot a_1 \end{aligned}$$

and by the substitutions $s = a - a_1 \in S$, $t = b - b_1 \in T$ and $a_2 = b_1 \cdot a_1 \in A$, we get that

$$\begin{aligned} b \cdot a - b_1 \cdot a_1 &= b \cdot (a - a_1) + (b - b_1) \cdot (b_1 \cdot a_1) - b_1 \cdot a_1 \\ &= b \cdot s + (t \cdot a_2) - a_2. \end{aligned}$$

Since $b \cdot s, (t \cdot a_2) - a_2 \in S$ by the conditions [NCM2] and [NCM3], we have that $b \cdot s + (t \cdot a_2) - a_2 = b \cdot a - b_1 \cdot a_1 \in S$. This means $[b \cdot a] = [b_1 \cdot a_1]$ in B/T . Moreover

$$\begin{aligned} b \star a - b_1 \star a_1 &= b \star a - b \star a_1 + b \star a_1 - b_1 \star a_1 \\ &= b \star (a - a_1) + (b - b_1) \star a_1 \end{aligned}$$

and by the same substitutions above we get

$$b \star a - b_1 \star a_1 = b \star s + t \star a_1.$$

and so by the conditions [NCM4] and [NCM5] $b \star a - b_1 \star a_1 \in S$. Hence $[b \star a] = [b_1 \star a_1]$ and therefore these actions are well defined.

These are derived actions since the conditions of [13, Proposition 1.1] are satisfied.

On the other hand it is clear that the boundary map, $\alpha^*: A/S \rightarrow B/T$ defined by $\alpha^*([a]) = [\alpha(a)]$, is well defined and the conditions [CM1]-[CM4] of Definition 3.10 are satisfied. ■

The following result is useful for some proofs (see for example the proofs of Theorem 5.11 and Theorem 6.7). The proof is clear and so it is omitted.

Proposition 3.16 *Let*

$$\begin{array}{ccccccc} 1 & \longrightarrow & S & \longrightarrow & A & \longrightarrow & P \longrightarrow 1 \\ & & \downarrow \sigma & & \downarrow \alpha & & \downarrow \pi \\ 1 & \longrightarrow & T & \longrightarrow & B & \longrightarrow & Q \longrightarrow 1 \end{array}$$

be a short exact sequence of crossed modules in \mathcal{C} . Then (S, T, σ) is a normal subcrossed module of

(A, B, α) and we have the short exact sequence of groups with operations

$$1 \longrightarrow S \rtimes T \longrightarrow A \rtimes B \longrightarrow P \rtimes Q \longrightarrow 1$$

and so $S \rtimes T$ is an ideal of $A \rtimes B$.

In the following result we prove that a normal subcrossed module in \mathbf{C} is categorically a normal object in the category of crossed modules in \mathbf{C} .

Theorem 3.17 *(S, T, σ) is a normal subcrossed module of (A, B, α) if and only if it is a kernel of a morphism $(f, g): (A, B, \alpha) \rightarrow (C, D, \gamma)$ of crossed modules in \mathbf{C} .*

Proof: We know that the kernel of a crossed module morphism $(f, g): (A, B, \alpha) \rightarrow (C, D, \gamma)$ is a normal subcrossed module of (A, B, α) .

Conversely let (S, T, σ) be a normal subcrossed module of (A, B, α) . Then we have a morphism $(p_A, p_B): (A, B, \alpha) \rightarrow (A/S, B/T, \alpha^*)$ of crossed modules whose kernel is the crossed module (S, T, σ) , where $p_A: A \rightarrow A/S$ and $p_B: B \rightarrow B/T$ are the natural projections. ■

Proposition 3.18 *Let (S, T, σ) be a normal subcrossed module of (A, B, α) in \mathbf{C} . Then the semi-direct groups with operations $A/S \rtimes B/T$ and $(A \rtimes B)/(S \rtimes T)$ are isomorph in \mathbf{C} .*

Proof: It can be proved that the function

$$\begin{aligned} \varphi : A/S \rtimes B/T &\longrightarrow (A \rtimes B)/(S \rtimes T) \\ ([a], [b]) &\longmapsto [(a, b)] \end{aligned}$$

is an isomorphism in \mathbf{C} . ■

The isomorphism theorem for crossed modules in \mathbf{C} can be given as follows.

Theorem 3.19 *Let $(f, g): (A, B, \alpha) \rightarrow (A', B', \alpha')$ be a morphism of crossed modules; and let $\text{Ker } f = S$ and $\text{Ker } g = T$. Then the image $(f(A), g(B), \alpha')$ is a subcrossed module and isomorph to the quotient crossed module $(A/S, B/T, \rho)$.*

Proof: It is easy to see that $(f(A), g(B), \alpha')$ is a subcrossed module and

$$(\tilde{f}, \tilde{g}): (A/S, B/T, \rho) \rightarrow (f(A), g(B), \alpha')$$

is an isomorphism of crossed modules, where $\tilde{f}(aS) = f(a)$ and $\tilde{g}(bT) = g(b)$ for $aS \in A/S$ and $bT \in B/T$. ■

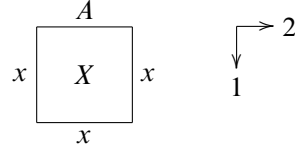
Corollary 3.20 *Let $(f, g): (A, B, \alpha) \rightarrow (A', B', \alpha')$ be an epimorphism of crossed modules; and let $\text{Ker } f = S$ and $\text{Ker } g = T$. Then the image (A', B', α') is isomorph to the quotient crossed module $(A/S, B/T, \rho)$.*

4 Fundamental crossed modules in group with operations

We recall a major geometric example of a crossed module over groups as follows: Let (X, A, x) be a based pair of spaces, where X is a topological space and $x \in A \subseteq X$. As Whitehead proved the boundary map

$$\partial: \pi_2(X, A, x) \rightarrow \pi_1(A, x)$$

from the second relative homotopy group of (X, A, x) to the fundamental group $\pi_1(A, x)$, together with the standard action of $\pi_1(A, x)$ on $\pi_2(X, A, x)$ has the structure of crossed module. Here the elements of the second relative homotopy group $\pi_2(X, A, x)$ are the homotopy classes of the relative paths as pictured below and the compositions in the both directions denoted induce the same group This is called *fundamental crossed module* of the based pair



(X, A, x) and denoted by $\Pi_2(X, A, x)$. In this manner we have a functor from based pairs of topological spaces to the crossed modules over groups

$$\text{Top}_*^2 \rightarrow \text{XMod}(\text{Grp}).$$

A 2-dimensional Seifert-van Kampen Theorem for fundamental crossed modules was proved in [6, Theorem 2.3.1].

We know from [25, Proposition 3.5] that if A is a topological group with operations in Top^C , then $\pi_1(A, 0)$ is a group with operations in C . We now prove that if X is a topological group with operations in Top^C and A is a subobject of A in C , then $\pi_2(X, A, 0)$ is also a group with operations and the boundary morphism

$$\partial: \pi_2(X, A, 0) \rightarrow \pi_1(A, 0)$$

becomes a crossed module of groups with operations in C .

As a preparation for the second relative paths α_1, α_2 and β_1, β_2 by the evaluation of the 2-dimensional paths we obtain an interchange rule

$$(\alpha_1 \circ \beta_1) + (\alpha_2 \circ \beta_2) = (\alpha_1 + \alpha_2) \circ (\beta_1 + \beta_2) \quad (3)$$

whenever the compositions of the paths $\alpha_1 \circ \beta_1$ and $\alpha_2 \circ \beta_2$ are defined. Hence by the interchange rule (3) for second relative paths α, β we have that

$$\alpha + \beta \simeq (\alpha \circ \alpha_0) + (\alpha_0 \circ \beta) = (\alpha + \alpha_0) \circ (\alpha_0 + \beta) = \alpha \circ \beta \quad (4)$$

where α_0 is the second zero path defined by $\alpha_0(s, t) = 0$ for $0 \leq s, t \leq 1$. Hence two group operations on $\pi_2(X, A, 0)$ are the same.

Theorem 4.1 *If X is a topological group with operations in Top^C and A is a subobject of X in C , then $\pi_2(X, A, 0)$ is a group with operations in C and it is abelian with respect to “+”.*

Proof: The binary operations on $\pi_2(X, A, 0)$ are defined by $[\alpha] \star [\beta] = [\alpha \star \beta]$ for $\star \in \Omega_2$ and the unary operations are defined by $\omega[\alpha] = [\omega(\alpha)]$ for $\omega \in \Omega_1$. The other details are satisfied and hence $\pi_2(X, A, 0)$ becomes a group with operations.

Let α and β be second relative paths. Then we define a homotopy

$$F(r, s, t) = -\alpha(r, st) + \alpha(r, s) + \beta(r, s) + \alpha(r, st). \quad (5)$$

Here $F(r, s, 0) = \alpha(r, s) + \beta(r, s)$ and $F(r, s, 1) = \beta(r, s) + \alpha(r, s)$. Hence $\alpha + \beta$ and $\beta + \alpha$ are homotopic. Therefore $\pi_2(X, A, 0)$ is abelian with respect to “+”. ■

Theorem 4.2 *If X is a topological group with operations in Top^C and A is a subobject of X in C , then the boundary morphism*

$$\partial: \pi_2(X, A, 0) \rightarrow \pi_1(A, 0)$$

becomes a crossed module of groups with operations in C .

Proof: By Theorem 4.1, $\pi_2(X, A, 0)$ is abelian with respect to “+” and for $[b] \in \pi_1(A, 0)$, $[\alpha] \in \pi_2(X, A, 0)$ the actions of $\pi_1(A, 0)$ on $\pi_2(X, A, 0)$ are defined by

$$\begin{aligned} [b] \cdot [\alpha] &= [\alpha] \\ [b] \star [a] &= [\beta \star a] \end{aligned}$$

for $\star \in \Omega'_2$ where $\beta(s, t) = b(s)$ for $0 \leq t \leq 1$. It is immediate to prove that these are well defined derived actions and the conditions of Definition 3.10 are satisfied. Therefore

$$\partial: \pi_2(X, A, 0) \rightarrow \pi_1(A, 0)$$

becomes a crossed module of groups with operations.

■

Hence we have a functor from based pairs of topological groups with operations to the crossed modules over groups with operations

$$(\text{Top}^{\mathbb{C}})_*^2 \rightarrow \text{XMod}(\mathbb{C})$$

Corollary 4.3 *If $(X, A, 0)$ is a based pair topological group with operations and $(Y, B, 0)$ is a normal subobject of $(X, A, 0)$, then the crossed module $\Pi_2(Y, B, 0)$ is a normal subcrossed module of $\Pi_2(X, A, 0)$ in \mathbb{C} .*

5 Normal and quotient groupoids in the internal groupoids

Definition 5.1 An internal category C in \mathbb{C} is a category in which the initial and final point maps $d_0, d_1: C_1 \rightrightarrows C_0$, the object inclusion map $\epsilon: C_0 \rightarrow C_1$ and the partial composition $\circ: C_{1d_0} \times_{d_1} C_1 \rightarrow C_1$, $(b, a) \mapsto b \circ a$ are the morphisms in the category \mathbb{C} .

Note that since ϵ is a morphism in \mathbb{C} , $\epsilon(0) = 0$ and that the composition \circ being a morphism in \mathbb{C} , implies that for all $a, b, c, d \in C$ and $\star \in \Omega_2$

$$(a \star b) \circ (c \star d) = (a \circ c) \star (b \circ d) \quad (6)$$

whenever one side makes sense. This is called the *interchange law*.

As an application of the interchange rule (6) in an internal category C for $a, b \in C_1$ such that $d_0(b) = d_1(a) = y$ we have the equality

$$\begin{aligned} b \circ a &= (b + 1_0) \circ (1_y + (-1_y + a)) \\ &= (b \circ 1_y) + (1_0 \circ (-1_y + a)) \\ &= b - 1_y + a. \end{aligned} \quad (7)$$

As another easy application we note that any internal category C in \mathbb{C} is an internal groupoid since given $a \in C$,

$$a^{-1} = 1_{d_1(a)} - a + 1_{d_0(a)} \quad (8)$$

satisfies $a^{-1} \circ a = 1_{d_1(a)}$, $a \circ a^{-1} = 1_{d_0(a)}$ and the map $C \rightarrow C$, $a \mapsto a^{-1}$ is also a morphism in \mathbb{C} .

In particular if \mathbb{C} is the category of groups, then an internal category C in \mathbb{C} becomes a group object in the category of groupoids, which is called as *group-groupoid*, *2-group* or

\mathcal{G} -groupoids [11].

Example 5.2 Let A be an object in \mathbf{C} . Then the groupoid $G = A \times A$ is an internal groupoid: Here a pair (a, b) is a morphism from a to b with inverse morphism (b, a) . The groupoid composition is defined by $(c, d) \circ (a, b) = (a, d)$ whenever $b = c$. The induced group operations are defined by $(a, b) \star (c, d) = (a \star c, b \star d)$ for $\star \in \Omega_2$ and $\omega(a, b) = (\omega(a), \omega(b))$ for $\omega \in \Omega_1$.

Example 5.3 [2, Example 3.8] If X is an object in $\mathbf{Top}^{\mathbf{C}}$, then the fundamental groupoid πX is an internal groupoid in \mathbf{C} .

Remark 5.4 We emphasize the following points from Definition 5.1 [2, Remark 3.7]:

- (i) By Definition 5.1 we know that in an internal groupoid G in \mathbf{C} , the initial and final point maps d_0 and d_1 , the object inclusion map ϵ are the morphisms in \mathbf{C} and the interchange law (6) is satisfied. Therefore in an internal groupoid G , the unary operations are endomorphisms of the underlying groupoid of G and the binary operations are morphisms from the underlying groupoid of $G \times G$ to the one of G .
- (ii) Let G be an internal groupoid in \mathbf{C} and $0 \in G_0$ the identity element. Then $\text{Ker} d_0 = \text{St}_G 0$, called in [4] *transitivity component* or *connected component* of 0 , is also an internal groupoid which is also an ideal of G .

Lemma 5.5 Let G be an internal groupoid in \mathbf{C} and N a wide subgroupoid of G . Then N is a normal subgroupoid of G in the sense of Definition 2.1 if and only if

$$-1_x + N(x) = -1_y + N(y)$$

for all $x, y \in G_0$ with $G(x, y) \neq \emptyset$.

Proof: Let N be a normal subgroupoid of G . Then $g \circ N(x) = N(y) \circ g$ or by Equation (7) equivalently

$$g - 1_x + N(x) = N(y) - 1_y + g$$

for $x, y \in G_0$ and $g \in G(x, y)$. Hence

$$g + (-1_x + N(x)) + (-g + 1_y) = N(y)$$

and here all morphisms of $-1_x + N(x)$ are in $\text{Ker} d_0$ and $-g + 1_y \in \text{Ker} d_1$. So morphisms of

$-1_x + N(x)$ commute with $-g + 1_y$. Therefore we can write

$$\begin{aligned} g - 1_x + N(x) &= N(y) - 1_y + g \\ g + (-g + 1_y) + (-1_x + N(x)) &= N(y) \\ 1_y - 1_x + N(x) &= N(y) \\ -1_x + N(x) &= -1_y + N(y). \end{aligned}$$

Conversely let

$$-1_x + N(x) = -1_y + N(y)$$

for all $x, y \in G_0$ with $G(x, y) \neq \emptyset$. By reversing the steps above we get $g \circ N(x) = N(y) \circ g$ for objects $x, y \in G_0$ and $g \in G(x, y)$. Hence N becomes a normal subgroupoid of G . ■

The details of the following theorem is given in [1].

Proposition 5.6 *Let G be an internal groupoid in the groups with operations in \mathcal{C} and N a normal subgroupoid of G . If N_1 is a subobject of G_1 in \mathcal{C} , then the quotient groupoid G/N becomes an internal groupoid in the groups with operations.*

The following theorem was proved in [29, Theorem1]. Since some details of the proof will be used in later proofs we give a sketch proof.

Theorem 5.7 *The category $\text{XMod}(\mathcal{C})$ of crossed modules and the category $\text{Cat}(\mathcal{C})$ of internal groupoids in \mathcal{C} are equivalent.*

Proof: A functor $\delta: \text{Cat}(\mathcal{C}) \rightarrow \text{XMod}(\mathcal{C})$ is defined as follows: For an internal groupoid G , let $\delta(G)$ be the crossed module (A, B, d_1) in \mathcal{C} , where $A = \text{Ker}d_0$, $B = G_0$ and $d_1: A \rightarrow B$ is the restriction of the target point map. Here A and B inherit the structures of group with operations from that of G , and the target point map $d_1: A \rightarrow B$ is a morphism in \mathcal{C} . Further the actions $B \times A \rightarrow A$ on the group with operations A are defined by

$$\begin{aligned} b \cdot a &= \epsilon(b) + a - \epsilon(b) \\ b \star a &= \epsilon(b) \star a \end{aligned}$$

for $a \in A$, $b \in B$. The axioms of Definition 3.10 are satisfied. Thus (A, B, d_1) becomes a crossed module in \mathcal{C} .

Conversely define a functor $\eta: \text{XMod}(\mathcal{C}) \rightarrow \text{Cat}(\mathcal{C})$ in the following way. For a crossed module (A, B, α) in \mathcal{C} , define an internal groupoid $\eta(A, B, \alpha)$ whose set of objects is the group with operations B and set of morphisms is the semi-direct product $A \rtimes B$ which is a group with operations by Theorem 3.8. The source and target point maps are defined to be

$d_0(a, b) = b$ and $d_1(a, b) = \alpha(a) + b$ while the object inclusion map and groupoid composition is given by $\epsilon(b) = (0, b)$ and

$$(a_1, b_1) \circ (a, b) = (a_1 + a, b)$$

whenever $b_1 = \alpha(a) + b$. Hence $\eta(A, B, \alpha)$ is an internal groupoid. The other details of the proof is obtained from that of [29, Theorem 1]. ■

Lemma 5.8 *Let G be an internal groupoid in \mathcal{C} and H a subgroupoid of G .*

(a) *If H_1 is a subobject of G_1 , then H_0 is also a subobject of G_0 .*

(b) *If H_1 is an ideal of G_1 then H_0 is also an ideal of G_0 .*

Proof:

(a) Let $x, y \in H_0$. Since H is a subgroupoid of G $1_x, 1_y \in H_1$ and since H_1 is a subobject of G_1 $1_x \star 1_y = 1_{x \star y} \in H_1$ and $\omega(1_x) = 1_{\omega(x)} \in H_1$. So $x \star y \in H_0$ and $\omega(x) \in H_0$. Hence H_0 is a subobject of G_0 .

(b) Let $x \in G_0$ and $y \in H_0$. In this case $1_x \in G_1$ and $1_y \in H_1$. Since H_1 is an ideal of G_1 we have that $1_x + 1_y - 1_x = 1_{x+y-x} \in H_1$ and $1_x \star 1_y = 1_{x \star y} \in H_1$ and so $x + y - x \in H_0$, i.e. $(H_0, +)$ is a normal subgroup of $(G_0, +)$, and $x \star y \in H_0$. Hence H_0 becomes an ideal of G_0 . ■

By Theorem 5.7 and Lemma 5.8 we relate the subobjects in these categories as follows:

Theorem 5.9 *Let (S, T, σ) be a subcrossed module of a crossed module (A, B, α) in \mathcal{C} . Suppose that H and G are respectively the internal groupoids in \mathcal{C} corresponding to these crossed modules. Then H_1 is a subobject of G_1 and H_0 is a subobject of G_0 .*

Proof: By the detail of the proof of Theorem 5.7, we know that $H_1 = S \rtimes T$ and $G_1 = A \rtimes B$. It is clear that $S \rtimes T$ is a subobject of $A \rtimes B$ since (S, T, σ) is a subcrossed module of (A, B, α) . Hence H_1 is a subobject of G_1 and by Lemma 5.8 H_0 is also a subobject of G_0 . ■

Hence the notion of internal subgroupoid of an internal groupoid in \mathcal{C} can be stated as follows:

Definition 5.10 Let G be an internal groupoid in groups with operations in \mathcal{C} and H a subgroupoid of G such that H_1 is a subobject of G_1 . Then H is called an *internal subgroupoid* of G .

By Theorem 5.7 and Lemma 5.8 we relate the normal subobjects in these categories as follows:

Theorem 5.11 *Let (S, T, σ) be a normal subcrossed module of a crossed module (A, B, α) in \mathcal{C} . Suppose that N and G are respectively the internal groupoids in \mathcal{C} corresponding to these crossed modules. Then H_1 is an ideal of G_1 and H_0 is also an ideal of G_0 .*

Proof: By the proof of Theorem 5.7 $N_1 = S \rtimes T$ and $G_1 = A \rtimes B$. By Proposition 3.16 $S \rtimes T$ is an ideal of $A \rtimes B$. Therefore H_1 is an ideal of G_1 and by Lemma 5.8 H_0 is an ideal of G_0 . ■

Hence the notion of internal normal subgroupoid of an internal groupoid in \mathcal{C} can be stated as follows:

Definition 5.12 Let G be an internal groupoid in groups with operations in \mathcal{C} and N a subgroupoid of G such that N_1 is an ideal of G_1 . Then N is called an *internal normal subgroupoid* of G .

Example 5.13 These are some examples of internal normal subgroupoids:

- (i) Let A be an object in \mathcal{C} and B an ideal of A . Then $N = B \times B$ as defined in Example 5.2 becomes an internal normal subgroupoid of $G = A \times A$.
- (ii) Let G and H be two internal groupoids in \mathcal{C} and $f: G \rightarrow H$ a morphism of internal groupoids. Then the kernel

$$\text{Ker } f = \{a \in G_1 \mid f(a) = 1_0 \in H_1\}$$

of f is an internal normal subgroupoid of G .

- (iii) If X is an object in $\text{Top}^{\mathcal{C}}$ and Y an ideal of X in \mathcal{C} , then the fundamental groupoid πY is an internal normal subgroupoid of πX .

Theorem 5.14 *Let G be an internal groupoid in \mathcal{C} and N an internal normal subgroupoid of G . Then the crossed module corresponding to N is a normal subcrossed module of the one corresponding to G .*

Proof: By Definition 5.12 N_1 is an ideal of G_1 . Let (A, B, α) and (S, T, σ) be the corresponding crossed modules to G and N respectively. So $A = \text{Ker } d_0$, $B = G_0$, $S = \text{Ker } d_0 \cap N_1$ and $T = N_0$. To prove that (S, T, σ) is a subcrossed module of (A, B, α) , we need to show that (S, T, σ) satisfies the conditions of Definition 3.12.

NCM1. We know that by Lemma 5.8 T is an ideal of B .

NCM2. Let $b \in B$ and $s \in S$. Then by the proof of Theorem 5.7 $b \cdot s = 1_b + s - 1_b$ where $1_b, -1_b \in G_1$. Since N_1 is an ideal of G_1 we have that $1_b + s - 1_b = b \cdot s \in N_1$ and

$$\begin{aligned} d_0(1_b + s - 1_b) &= d_0(1_b) + d_0(s) - d_0(1_b) \\ &= b + 0 - b \\ &= 0. \end{aligned}$$

Hence $b \cdot s \in S$.

NCM3. Let $t \in T$ and $a \in A$. Then

$$\begin{aligned} d_0((t \cdot a) - a) &= d_0(1_t + a - 1_t - a) \\ &= d_0(1_t) + d_0(a) - d_0(1_t) - d_0(a) \\ &= t + 0 - t - 0 \\ &= 0 \end{aligned}$$

and so $(t \cdot a) - a \in A$. Moreover since $a \in G_1$, $1_t \in N_1$ and N_1 is an ideal of G_1 it follows that $a - 1_t - a \in N_1$ and since $1_t \in N_1$ it implies that $1_t + a - 1_t - a = (t \cdot a) - a \in N_1$. So $(t \cdot a) - a \in S$.

NCM4. Let $b \in B$ and $s \in S$. Then $b \star s = 1_b \star s \in N_1$ since $1_b \in G_1$, $s \in N_1$ and N_1 is an ideal of G_1 . Also

$$\begin{aligned} d_0(b \star s) &= d_0(1_b \star s) \\ &= d_0(1_b) \star d_0(s) \\ &= b \star 0 \\ &= 0 \end{aligned}$$

and so $b \star s \in A = \text{Ker}d_0$. Thus $b \star s \in S$.

NCM5. Let $t \in T$ and $a \in A$. Then $t \star a = 1_t \star a \in N_1$ since $1_t \in N_1$, $a \in G_1$ and N_1 is an

ideal of G_1 . Further

$$\begin{aligned}
 d_0(t \star a) &= d_0(1_t \star a) \\
 &= d_0(1_t) \star d_0(a) \\
 &= t \star 0 \\
 &= 0
 \end{aligned}$$

and so $t \star a = 1_t \star a \in A = \text{Ker} d_0$. Thus $t \star a \in S$

Therefore (S, T, σ) is a normal subcrossed module of (A, B, α) . ■

As a result of Theorem 5.11 and Theorem 5.14 we can state the following corollary.

Corollary 5.15 *Let G be an internal groupoid in \mathcal{C} and (A, B, μ) the crossed module corresponding to G . Then the category $\text{NSGd}_{\text{Cat}(\mathcal{C})}/G$ of internal normal subgroupoids of G and the category $\text{NSCM}/(A, B, \mu)$ of normal subcrossed modules of (A, B, μ) are equivalent.*

We now define internal quotient groupoid as follows.

Definition 5.16 Let G be an internal groupoid in \mathcal{C} and N an internal normal subgroupoid of G . Let (A, B, α) and (S, T, σ) be respectively the crossed modules corresponding to G and N . Then the internal groupoid corresponding to the quotient crossed module $(A/S, B/T, \rho)$ is called an *internal quotient groupoid* and denoted by G_N .

So the set of objects and the set of morphisms of G_N are respectively the quotient groups with operations G_0/N_0 and G_1/N_1 .

Here note that in the internal quotient groupoid G_N , the internal normal subgroupoid N is not wide and so is not a normal subgroupoid in the sense of Higgins [17] and Brown [4]. If N is wide then G_N becomes a singular object, i.e. G_N is an abelian group and $[g] \star [g_1] = 0$ for all $[g], [g_1] \in G_N$, in \mathcal{C} . Therefore the internal quotient groupoid G_N defined in Definition 5.16 is not a quotient groupoid G/N in the sense of Higgins [17].

In the following theorem we compare and relate quotient groupoid and internal quotient groupoid.

Theorem 5.17 *Let G be an internal groupoid in \mathcal{C} and N an internal normal subgroupoid of G . If N is wide in G , then N is a normal subgroupoid of G and also the quotient groupoid G/N and the internal quotient groupoid G_N are same if and only if N is transitive.*

Proof: Let $g \in G(x, y)$ and $n \in N(x)$. Then

$$\begin{aligned} g \circ n &= n - 1_x + g \\ &= n - 1_x + 1_y - 1_y + g \\ &= n' - 1_y + g \\ &= n' \circ g \in N(y) \circ g \end{aligned}$$

where $n' = n - 1_x + 1_y \in N(y)$ since N_1 is an ideal of G_1 . Hence $g \circ N(x) \subset N(y) \circ g$ and similarly $N(y) \circ g \subset g \circ N(x)$. Therefore if N is wide, then it is a normal subgroupoid of G .

Let the quotient groupoid G/N and the internal quotient groupoid G_N be same. Since N is wide in G then G_N , and hence G/N , has only one object. Hence N is transitive.

On the other hand there are two equivalence relations on the set of morphisms G_1 .

$g \sim_1 g'$ if and only if there exists $n \in N_1$ such that $g = g' + n$. Thus G_1 / \sim_1 gives the quotient by the group with operations structure.

$g \sim_2 g'$ if and only if there are arrows $m, n \in N_1$ such that $g = n \circ g' \circ m$. Thus G_1 / \sim_2 gives the groupoid quotient.

If N is transitive, then these two groupoids have only one object, i.e, they are singular objects in \mathcal{C} . Suppose that $g \sim_2 g'$ in G_1 so that there are $m, n \in N_1$ such that $g = n \circ g' \circ m$. Then also

$$\begin{aligned} g &= n \circ g' \circ m \\ &= n - 1_{d_0(n)} + g' - 1_{d_0(g')} + m \\ &= g' - g' + n - 1_{d_0(n)} + g' - 1_{d_0(g')} + m \\ &= g' + (-g' + n - 1_{d_0(n)} + g' - 1_{d_0(g')} + m). \end{aligned}$$

Since N is an internal normal subgroupoid of G ,

$$(-g' + n - 1_{d_0(n)} + g' - 1_{d_0(g')} + m) \in N_1$$

so writing n_1 for $(-g' + n - 1_{d_0(n)} + g' - 1_{d_0(g')} + m)$ it follows that $g = g' + n_1$ and hence $g \sim_1 g'$.

On the other hand, if $g \sim_1 g'$, then there is an $n \in N_1$ such that $g = g' + n$. Suppose $g \in G(y, v)$ and $n \in N(x, u)$. Then $g \in G(y + x, v + u)$. Since N is wide in G , then $1_x \in N_1$ for all $x \in G_0$. Since N is transitive then for each $x \in N_0$ we can choose a morphism $0_x \in N(0, x)$. Hence

$$[(1_v + n) \circ (1_v + 0_x)] \circ g' \circ (1_y + 0_x^{-1}) = g$$

This implies that $g \sim_2 g'$. Hence the quotient groupoid G/N and the internal quotient groupoid G_N are the same. ■

The following result characterize the normal objects in the category of internal groupoids in \mathcal{C} .

Theorem 5.18 *Let G and N be two internal groupoids in \mathcal{C} . Then N is an internal normal subgroupoid of G if and only if there exist an internal groupoid morphism $f: G \rightarrow H$ such that $\text{Ker} f = N$.*

Proof: We know that the kernel of a morphism $f: G \rightarrow H$ internal groupoids is an internal normal subgroupoid of G .

Conversely if N is an internal normal subgroupoid of G , then we can obtain the internal quotient groupoid G_N and quotient morphism $p: G \rightarrow G_N$ of internal groupoids in \mathcal{C} such that $\text{Ker} p = N$. ■

6 Normal and quotient cat^1 -groups with operations

We now generalize the notion of cat^1 -group to the groups with operations as follows.

Definition 6.1 Let A be an object in \mathcal{C} and s, t two endomorphisms of A in \mathcal{C} . If the following are satisfied for all $\star \in \Omega'_2$, then (A, s, t) is called a cat^1 -group with operations or cat^1 -group object in \mathcal{C} .

- (i) $st = t, ts = s$;
- (ii) $[\text{Ker} s, \text{Ker} t] = 0$;
- (iii) $\text{Ker} s \star \text{Ker} t = 0$.

Example 6.2 Here are some examples of cat^1 -groups with operations:

- (i) An object A in \mathcal{C} can be regarded as a cat^1 -group with operations by the endomorphisms $s = 1_A = t$.
- (ii) A singular object A in \mathcal{C} , i.e., an abelian group such that $a \star a' = 0$ for all $a, a' \in A$ and $\star \in \Omega'_2$, is a cat^1 -group with operations where s and t are zero morphisms.
- (iii) If B is an object of \mathcal{C} , then $A = B \times B$ is a cat^1 -group with operations by the endomorphisms s and t defined by $s(b, b_1) = (b, b)$ and $t(b, b_1) = (b_1, b_1)$.

Let (A, s, t) and (A', s', t') be two cat^1 -groups with operations. A morphism $f: A \rightarrow A'$ in \mathcal{C} is called a *morphism* of cat^1 -groups with operations if it is compatible with the endomorphisms s and t , i.e. if $fs = s'f$ and $ft = t'f$.

Hence we can construct a category denoted $\text{Cat}^1(\mathcal{C})$ of cat^1 -groups with operations and their morphisms. We now generalize a result proved in cat^1 -group case by Loday in [21] to the cat^1 -groups with operations in \mathcal{C} .

Theorem 6.3 *The category $\text{XMod}(\mathcal{C})$ of crossed modules and the category $\text{Cat}^1(\mathcal{C})$ of cat^1 -groups with operations in \mathcal{C} are equivalent.*

Proof: We give a sketch proof based on that of group case given in [21]. First of all define a functor

$$\delta: \text{Cat}^1(\mathcal{C}) \longrightarrow \text{XMod}(\mathcal{C})$$

as follows: For an object (A, s, t) of $\text{Cat}^1(\mathcal{C})$, $\delta(A, s, t)$ is $(\text{Ker}s, \text{Im}s, t|_{\text{Ker}s})$ which is an object of $\text{XMod}(\mathcal{C})$ where the actions of $\text{Im}s$ on $\text{Ker}s$ are

$$\begin{aligned} s(a) \cdot a_1 &= s(a) + a_1 - s(a) \\ s(a) \star a_1 &= s(a) \star a_1 \end{aligned}$$

for $a, a_1 \in A$ and $\star \in \Omega'_2$. If $f: (A, s, t) \rightarrow (A', s', t')$ is a morphism in $\text{Cat}^1(\mathcal{C})$, then

$$\delta(f) = (f|_{\text{Ker}s}, f|_{\text{Im}s}): (\text{Ker}s, \text{Im}s, t|_{\text{Ker}s}) \rightarrow (\text{Ker}s', \text{Im}s', t'|_{\text{Ker}s'})$$

is a morphism in $\text{XMod}(\mathcal{C})$.

Conversely define a functor

$$\theta: \text{XMod}(\mathcal{C}) \longrightarrow \text{Cat}^1(\mathcal{C})$$

in the following way: Let (A, B, α) be an object of $\text{XMod}(\mathcal{C})$. Then $\theta(A, B, \alpha) = (A \rtimes B, s, t)$ where $s(a, b) = (0, b)$ and $t(a, b) = (0, \alpha(a) + b)$. If $(f, g): (A, B, \alpha) \rightarrow (A', B', \alpha')$ is a morphism in $\text{XMod}(\mathcal{C})$, then $f \times g: (A \rtimes B, s, t) \rightarrow (A' \rtimes B', s', t')$ is a morphism in $\text{Cat}^1(\mathcal{C})$.

Moreover we obtain a natural equivalence $\eta: 1_{\text{XMod}(\mathcal{C})} \rightarrow \delta\theta$, where if (A, B, α) is a crossed module in \mathcal{C} , then $\eta_{(A, B, \alpha)}$ is given by $a \mapsto (a, 0)$ on A and by $b \mapsto (0, b)$ on B . Another natural equivalence $\mu: \theta\delta \rightarrow 1_{\text{Cat}^1(\mathcal{C})}$ is defined as follows: Let (A, s, t) be an object of $\text{Cat}^1(\mathcal{C})$. A morphism $\mu_{(A, s, t)}: \theta\delta((A, s, t)) \rightarrow (A, s, t)$ is given by $(a, s(b)) \mapsto a + s(b)$. It is easy to verify that this morphism is an isomorphism in $\text{Cat}^1(\mathcal{C})$. The rest of the proof is straightforward. ■

As a result of Theorem 5.7 and Theorem 6.3 we obtain the following corollary.

Corollary 6.4 *The category $\text{Cat}^1(\mathcal{C})$ of cat^1 -groups in \mathcal{C} and the category $\text{Cat}(\mathcal{C})$ of internal groupoids in \mathcal{C} are equivalent.*

By Theorem 6.3 we relate the subobjects in these categories as follows:

Theorem 6.5 *Let (S, T, σ) be a subcrossed module of a crossed module (A, B, α) in \mathcal{C} . Suppose that (H, s_H, t_H) and (G, s_G, t_G) are respectively the cat^1 -groups with operations corresponding to these crossed modules. Then H is a subobject of G ; and s_H and t_H are respectively the restrictions of s_G and t_G .*

Proof: By the detailed proof of Theorem 6.3, we know that $H = S \rtimes T$ and $G = A \rtimes B$; and s_H and t_H are respectively the restrictions of s_G and t_G . Since (S, T, σ) is a subcrossed module of (A, B, α) in \mathcal{C} , it follows that $S \rtimes T$ is a subobject of $A \rtimes B$ in \mathcal{C} . ■

Hence we can state the notion of subobjects in $\text{Cat}^1(\mathcal{C})$ as follows:

Definition 6.6 Let (G, s_G, t_G) and (H, s_H, t_H) be two objects of $\text{Cat}^1(\mathcal{C})$. If H is a subobject of G in \mathcal{C} ; and s_H and t_H are respectively the restrictions of s_G and t_G to H , then (H, s_H, t_H) is called a *subcat¹-group with operations* or *subobject* of (G, s_G, t_G) in $\text{Cat}^1(\mathcal{C})$.

Similarly by Theorem 6.3 we relate the normal subobjects in these categories as follows:

Theorem 6.7 *Let (S, T, σ) be a normal subcrossed module of a crossed module (A, B, α) in \mathcal{C} . Suppose that (N, s_N, t_N) and (G, s_G, t_G) are respectively the cat^1 -groups with operations corresponding to these crossed modules. Then (N, s_N, t_N) is an ideal of G .*

Proof: By the proof of Theorem 6.3, we know that $N = S \rtimes T$ and $G = A \rtimes B$ and N is an ideal of G by the Proposition 3.16. ■

Hence we can state the notion of normal subobject in $\text{Cat}^1(\mathcal{C})$ as follows:

Definition 6.8 Let (G, s_G, t_G) and (N, s_N, t_N) be two objects in $\text{Cat}^1(\mathcal{C})$. If N is an ideal of G in \mathcal{C} ; and s_N and t_N are respectively the restrictions of s_G and t_G to N , then (N, s_N, t_N) is called a *normal subcat¹-group with operations* or *normal subobject* of (G, s, t) in $\text{Cat}^1(\mathcal{C})$.

For example if $f: (G, s_G, t_G) \rightarrow (H, s_H, t_H)$ is a morphism of cat^1 -groups with operations in $\text{Cat}^1(\mathcal{C})$, then the kernel

$$\text{Ker } f = \{g \in G \mid f(g) = 0_H \in G\}$$

of f along with the endomorphisms $s|_{\text{Ker } f}$ and $t|_{\text{Ker } f}$ is a normal object of (G, s_G, t_G) .

Theorem 6.9 Let (G, s_G, t_G) be an object in $\text{Cat}^1(\mathcal{C})$ and (N, s_N, t_N) be a normal subcat¹-group with operations of (G, s_G, t_G) . If (A, B, α) and (S, T, σ) are respectively the crossed modules corresponding to (G, s_G, t_G) and (N, s_N, t_N) . Then (S, T, σ) is a normal subcrossed module of (A, B, α) in \mathcal{C} .

Proof: By the proof of Theorem 6.3 we have that $A = \text{Ker}s_G$, $B = \text{Im}s_G$, $S = \text{Ker}s_N = A \cap N$ and $T = \text{Im}s_N = s(N)$. We need to show that (S, T, σ) satisfies the conditions of Definition 3.12.

NCM1. If $g \in G$ and $n \in N$, then $s(g) + s(n) - s(g) = s(g + n - g) \in s(N) = T$ and $s(g) \star s(n) = s(g \star n) \in s(N) = T$ since N is an ideal of G . Hence T is an ideal of B .

NCM2. If $s(g) \in B$ and $n \in S$, then by the proof of Theorem 6.3 $s(g) \cdot n = s(g) + n - s(g) \in N$ since N is an ideal of G . Further

$$\begin{aligned} s(s(g) \cdot n) &= s(s(g) + n - s(g)) \\ &= s(s(g)) + s(n) - s(s(g)) \\ &= s(g) + 0 - s(g) \\ &= 0. \end{aligned}$$

and therefore $s(g) \cdot n \in S$.

NCM3. For $s(n) \in T$ and $g \in A$ we have that

$$\begin{aligned} s((s(n) \cdot g) - g) &= s(s(n) + g - s(n) - g) \\ &= s(s(n)) + s(g) - s(s(n)) - s(g) \\ &= s(n) + 0 - s(n) - 0 \\ &= 0 \end{aligned}$$

and so $(s(n) \cdot g) - g \in A$. Since $g \in G$, $s(n) \in N$ and N is an ideal of G these imply $g - s(n) - g \in N$ and since $s(n) \in N$ it implies that $s(n) + g - s(n) - g = (s(n) \cdot g) - g \in N$. Hence $(s(n) \cdot g) - g \in S$.

NCM4. Let $s(g) \in B$ and $n \in S$. Then by the proof of Theorem 6.3 $s(g) \star n = s(g) \star n \in N$

since N is an ideal of G . Further

$$\begin{aligned} s(s(g) \star n) &= s(s(g) \star n) \\ &= s(s(g)) \star s(n) \\ &= s(g) \star 0 \\ &= 0 \end{aligned}$$

and thus $s(g) \star n \in S$.

NCM5. For $s(n) \in T$ and $g \in A$ we have

$$\begin{aligned} s(s(n) \star g) &= s(s(n) \star g) \\ &= s(s(n)) \star s(g) \\ &= s(n) \star 0 \\ &= 0 \end{aligned}$$

and so $s(n) \star g \in A$. Since $g \in G$, $s(n) \in N$ and N is an ideal of G it follows that $s(n) \star g \in N$. Hence $s(n) \star g \in S$.

Therefore (S, T, σ) is a normal subcrossed module of (A, B, α) . ■

We now can construct the quotient objects in the category $\text{Cat}^1(\mathbf{C})$ as follows: Let (G, s_G, t_G) be an object in $\text{Cat}^1(\mathbf{C})$ and (N, s_N, t_N) a normal subobject of (G, s_G, t_G) . Then $(G/N, s_{G/N}, t_{G/N})$ becomes a cat^1 -group with operations by the induced endomorphisms in \mathbf{C}

$$\begin{aligned} s_{G/N}([g]) &= [s_G(g)] \\ t_{G/N}([g]) &= [t_G(g)]. \end{aligned}$$

This cat^1 -group with operations is called the *quotient cat^1 -group with operations* of (G, s_G, t_G) by (N, s_N, t_N) .

In the following theorem we prove that normal sub cat^1 -groups with operations are really the normal objects in the category $\text{Cat}^1(\mathbf{C})$.

Theorem 6.10 *Let (G, s_G, t_G) and (N, s_N, t_N) be two cat^1 -groups with operations. Then (N, s_N, t_N) is a normal sub cat^1 -group with operations of (G, s_G, t_G) if and only if it is a kernel of some morphism $f: (G, s_G, t_G) \rightarrow (H, s_H, t_H)$ of cat^1 -groups with operations.*

Proof: We know the kernel of a morphism $f: (G, s_G, t_G) \rightarrow (H, s_H, t_H)$ in $\text{Cat}^1(\mathbf{C})$ is a normal sub cat^1 -group with operations of (G, s_G, t_G) .

Conversely if (N, s_N, t_N) is a normal subcat¹-group with operations of (G, s_G, t_G) , then $p: (G, s_G, t_G) \rightarrow (G/N, s_{G/N}, t_{G/N})$ is a morphism in $\text{Cat}^1(\mathbb{C})$ and $\text{Ker} p = (N, s_N, t_N)$. ■

7 Covering morphisms of internal groupoids and cat¹-groups with operations in \mathbb{C}

Let \mathbb{C} be an arbitrary category with pullbacks. The notion of *covering morphism* in \mathbb{C} is defined in [8, pp.145] as a morphism $p: \tilde{G} \rightarrow G$ of internal groupoids in \mathbb{C} such that

$$(p_1, d_0): \tilde{G}_1 \longrightarrow G_{1d_0} \times_{p_0} \tilde{G}_0$$

is an isomorphism in \mathbb{C} . Equivalently it is defined in [2] as an internal groupoid morphism $p: \tilde{G} \rightarrow G$ in \mathbb{C} such that p is a covering morphism on the underlying groupoids.

Using the equivalence of the categories in Theorem 5.7, in [2] a *covering morphism* of crossed modules is defined as a morphism $(f, g): (\tilde{A}, \tilde{B}, \tilde{\alpha}) \rightarrow (A, B, \alpha)$ of crossed modules in \mathbb{C} such that if $f: \tilde{A} \rightarrow A$ is an isomorphism in \mathbb{C} .

We now give a result for coverings of internal quotient groupoids.

Proposition 7.1 *Let $p: \tilde{G} \rightarrow G$ be a covering morphism in \mathbb{C} , N an internal normal subgroupoid of G and $\tilde{N} = p^{-1}(N)$. Then \tilde{N} is an internal normal subgroupoid of \tilde{G} and the induced morphism $p_*: \tilde{G}_{\tilde{N}} \rightarrow G_N$ is a covering morphism in \mathbb{C} .*

Proof: It is straightforward to prove that $\tilde{N} = p^{-1}(N)$ is an internal normal subgroupoid of \tilde{G} . Let $(f, g): (\tilde{A}, \tilde{B}, \tilde{\alpha}) \rightarrow (A, B, \alpha)$ be the morphism of crossed modules corresponding to p . Since p is a covering morphism of internal groupoids in \mathbb{C} then $f: \tilde{A} \rightarrow A$ is an isomorphism in \mathbb{C} . Therefore $f_*: \tilde{A}/\tilde{S} \rightarrow A/S$ is an isomorphism in \mathbb{C} where $\tilde{S} = \text{Ker} d_{0|\tilde{N}_1}$ and $S = \text{Ker} d_{0|N_1}$. Hence $p_*: \tilde{G}_{\tilde{N}} \rightarrow G_N$ is a covering morphism in \mathbb{C} . ■

Let G an internal groupoid in \mathbb{C} and (A, B, α) be the crossed module corresponding to G by Theorem 5.7. Then it is proved in [2] that the category $\text{Cov}_{\text{Cat}(\mathbb{C})}/G$ of covering morphisms based on G and the category $\text{Cov}_{\text{XMod}(\mathbb{C})}/(A, B, \alpha)$ of covering morphisms of crossed module based on (A, B, α) are equivalent.

We now give a parallel result for cat¹-groups with operations.

Definition 7.2 A morphism $p: (\tilde{G}, s_{\tilde{G}}, t_{\tilde{G}}) \rightarrow (G, s_G, t_G)$ of cat¹-groups with operations is called a *covering morphism* in $\text{Cat}^1(\mathbb{C})$ if the restriction $p|_{\text{Ker} s_{\tilde{G}}}: \text{Ker} s_{\tilde{G}} \rightarrow \text{Ker} s_G$ of p to $\text{Ker} s_{\tilde{G}}$ is an isomorphism in \mathbb{C} .

Let $\text{Cov}_{\text{Cat}^1(\mathbf{C})}/(G, s_G, t_G)$ be the category of covering morphisms in $\text{Cat}^1(\mathbf{C})$ based on a cat^1 -group with operations (G, s_G, t_G) .

We finally give the following Theorem.

Theorem 7.3 *Let (A, B, α) be a crossed module and let (G, s_G, t_G) be the corresponding cat^1 -group with operations. Then the category $\text{Cov}_{\text{XMod}(\mathbf{C})}/(A, B, \alpha)$ of covers of (A, B, α) and the category $\text{Cov}_{\text{Cat}^1(\mathbf{C})}/(G, s_G, t_G)$ are equivalent.*

Proof: A functor $\eta: \text{Cov}_{\text{XMod}(\mathbf{C})}/(A, B, \alpha) \rightarrow \text{Cov}_{\text{Cat}^1(\mathbf{C})}/(G, s_G, t_G)$ is defined as follows: Let $(f, g): (\tilde{A}, \tilde{B}, \tilde{\alpha}) \rightarrow (A, B, \alpha)$ be an object in $\text{Cov}_{\text{XMod}(\mathbf{C})}/(A, B, \alpha)$. Then $\eta(f, g) = f \times g: \tilde{A} \rtimes \tilde{B} \rightarrow A \rtimes B$, $s_{\tilde{A} \rtimes \tilde{B}}(\tilde{a}, \tilde{b}) = (0, \tilde{b})$ and $s_{A \rtimes B}(a, b) = (0, b)$. Here $\text{Ker}s_{\tilde{A} \rtimes \tilde{B}} = \tilde{A} \times \{0\}$, $\text{Ker}s_{A \rtimes B} = A \times \{0\}$ and the restriction of $f \times g$ to $\text{Ker}s_{\tilde{A} \rtimes \tilde{B}}$ is $f \times 0$. Since (f, g) is a covering morphism then f and hence $f \times 0$ is an isomorphism in \mathbf{C} . Thus $\eta(f, g) = f \times g$ becomes an object in $\text{Cov}_{\text{Cat}^1(\mathbf{C})}/(G, s_G, t_G)$.

Conversely a functor $\theta: \text{Cov}_{\text{Cat}^1(\mathbf{C})}/(G, s_G, t_G) \rightarrow \text{Cov}_{\text{XMod}(\mathbf{C})}/(A, B, \alpha)$ is defined in the following way: Let $p: (\tilde{G}, s_{\tilde{G}}, t_{\tilde{G}}) \rightarrow (G, s_G, t_G)$ be an object in $\text{Cov}_{\text{Cat}^1(\mathbf{C})}/(G, s_G, t_G)$. Then $\theta(p) = (p|_{\text{Ker}s_{\tilde{G}}}, p|_{\text{Im}s_{\tilde{G}}})$ and clearly $\theta(p)$ is an isomorphism and hence an object in $\text{Cov}_{\text{XMod}(\mathbf{C})}/(A, B, \alpha)$.

The other details of the equivalence of the categories can be checked. ■

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